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Exact diffusion constant for the one-dimensional partially asymmetric exclusion model

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Abstract. We calculate exactly the diffusion constant associated with the fluctuations of the current for the partial asymmetric exclusion model on a ring with an arbitrary number of particles and holes. We also give the diffusion constant of a tagged particle on that ring. Our approach extends, using the deformed harmonic oscillator algebra, a result already known for the fully asymmetric case. In the limit of weak asymmetry, we extract from our exact expression the crossover between the Edwards–Wilkinson and the Kardar–Parisi–Zhang equations in $(1+1)$ dimensions.

1. Introduction

The one-dimensional asymmetric exclusion process (ASEP) is a lattice version of the Kardar–Parisi–Zhang (KPZ) equation [1–3]. It has been extensively studied by mathematicians [4–9] and physicists [10–20] as one of the simplest examples of a system out of equilibrium. The process describes particles that hop independently with hard-core exclusion along a one-dimensional lattice with a bias which mimics an external driving force. It is a simple case of a driven lattice gas [21] related to the hopping conductivity of superionic conductors [22] and to queuing problems [23]. One can also map it on the problem of directed polymers in a random medium [2, 3] in $(1+1)$ dimensions.

In the steady state, all the configurations of the exclusion model on a periodic lattice have equal probabilities [24]. This corresponds to the fact that the stationary measure of the one-dimensional KPZ equation is the Brownian 'free-field' measure [25]. The calculations of equal time correlation functions in the steady state are therefore very easy.

Unequal time properties, even in the steady state, are much more difficult to obtain [26, 27]. The gap between the largest two eigenvalues of the master equation that governs the time evolution of the ASEP has been computed by the Bethe ansatz [28–30] giving $3/2$ for the dynamical exponent of the one-dimensional KPZ equation. A different approach based on a matrix ansatz [31–34], initially used for the steady state of systems with open boundaries, has been extended to obtain some unequal time correlation functions like the diffusion constant of tagged particles on a ring [35] or the fluctuations of the current of a chain with open boundaries [36].

In the present paper we give the exact expression of the fluctuations of the total current through a bond for the partially asymmetric exclusion process on a ring. These fluctuations grow linearly with time and the constant of proportionality Δ can be interpreted as a diffusion constant. Up to a simple factor, Δ is also the diffusion constant of a tagged particle on a ring.

Our formula of Δ gives in particular the scaling function describing the crossover between the linear Edwards–Wilkinson (EW) model [37] and the nonlinear KPZ equation for the fluctuations of the height of a growing interface.

Let us first recall the dynamics of the one-dimensional exclusion model on a periodic lattice of L sites with P particles and H holes (with $L = P + H$). Each site i ($1 \leqslant i \leqslant L$) is either occupied by a particle ($\tau_i = 1$) or is empty ($\tau_i = 0$). The lattice has periodic boundary conditions meaning that $i \equiv i + L$. The system evolves according to a stochastic dynamical rule: during each infinitesimal time step dt, the only transitions allowed for the bond $(i, i+1)$ are

$$\begin{aligned} &10 \to 01 \qquad \text{with rate } 1 \\ &01 \to 10 \qquad \text{with rate } x. \end{aligned} \tag{1}$$

The parameter x is positive and it measures the strength of the driving force. For $x = 1$ the system is symmetric whereas for $x = 0$ (or $x = \infty$) it reduces to the totally asymmetric case [35].

In the long time limit, the system reaches a steady state in which all configurations $\mathcal{C}$ have the same weight [24]

$$p(\mathcal{C}) = \binom{P+H}{P}^{-1} = \frac{P!H!}{(P+H)!}. \tag{2}$$

Here we consider the current through a 'marked' bond, for instance the bond $(L, 1)$. If Y_t is the net number of particles having crossed that bond between time 0 and time t (i.e. the number of particles having crossed that bond from left to right *minus* those having crossed from right to left),

$$\lim_{t\to\infty} \frac{1}{t}\langle Y_t\rangle \to J \tag{3}$$

where the expression of the steady state current J follows easily from (2)

$$J = (1-x)\frac{PH}{(P+H)(P+H-1)}. \tag{4}$$

In a similar way one expects (and in fact one can prove [36]) that in the long time limit

$$\frac{\langle Y_t^2\rangle - \langle Y_t\rangle^2}{t} \to \Delta. \tag{5}$$

The main result of the present paper is the following exact formula

$$\Delta = 2\frac{(1-x)}{L(L-1)}\sum_{n=1}^{\infty} n^2 \frac{1+x^n}{1-x^n}\frac{(P!)^2(H!)^2}{(P+n)!(P-n)!(H+n)!(H-n)!} \tag{6}$$

for the fluctuations of the current through a bond on a ring of L sites with P particles and $H = L - P$ holes (here, we use the convention that $k! = \infty$ for $k \leqslant -1$ so that the sum (6) has only a finite number of terms with no contribution for $n > \min(P, H)$). Sections 2, 3 and 4 are devoted to the derivation of (6) and in section 5, we will see how this expression (6) leads for x close to 1 to the crossover between the symmetric process and the asymmetric process, i.e. between the EW and the KPZ equations [30, 38].

The knowledge of Δ also gives the diffusion constant of a tagged particle on a ring. If one calls X_t the position of a tagged particle (which is, in all respects, equivalent to the other $P-1$ particles), this position fluctuates about its average $[\langle X_t^2\rangle - \langle X_t\rangle^2 \simeq \Delta_{\text{tag}} t]$ and the corresponding diffusion constant Δ_{tag} is related to Δ by

$$\Delta_{\text{tag}} = \Delta \left(\frac{H+P}{P}\right)^2 .$$

This follows from the fact that, due to the hard core repulsion, all particles perform exactly the same number of rotations $\simeq Y_t/P$ and that each time a particle performs a rotation it covers a distance $H+P$.

2. The master equation

In this section, we recall [35, 36] how, in the calculation of Δ, the master equation which governs the dynamics can be reduced to a system of $\binom{P+H}{P}$ inhomogeneous linear equations.

Let us denote by $P_t(\mathcal{C}, Y)$ the probability of finding at time t the system in a configuration $\mathcal{C}$ of particles and with $Y_t = Y$ (Y_t is the algebraic number of particles having gone through the marked bond between time 0 and time t). The master equation that governs the time evolution of $P_t(\mathcal{C}, Y)$ has the following form

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t} P_t(\mathcal{C}, Y) = \sum_{\mathcal{C}'} M_0(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}', Y) + M_1(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}', Y-1) + M_{-1}(\mathcal{C}, \mathcal{C}') P_t(\mathcal{C}', Y+1) \\ - \left[\sum_{\mathcal{C}'} M_0(\mathcal{C}', \mathcal{C}) + M_1(\mathcal{C}', \mathcal{C}) + M_{-1}(\mathcal{C}', \mathcal{C})\right] P_t(\mathcal{C}, Y) \qquad (7)\end{aligned}$$

where $M_0(\mathcal{C}, \mathcal{C}')$ is the rate of transition from a configuration $\mathcal{C}$ to a configuration $\mathcal{C}'$ obtained by moving a particle that does not cross the marked bond, $M_1(\mathcal{C}, \mathcal{C}')$ (respectively $M_{-1}(\mathcal{C}, \mathcal{C}')$) is the rate of transition from a configuration $\mathcal{C}$ to a configuration $\mathcal{C}'$ obtained by moving a particle that does cross the marked bond in the positive (respectively negative) direction.

Multiplying both sides of (7) by Y or Y^2 and summing over $\mathcal{C}$ and Y leads to the evolution equation for the first two moments of Y_t

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Y_t\rangle = \sum_{\mathcal{C},\mathcal{C}'} [M_1(\mathcal{C}, \mathcal{C}') - M_{-1}(\mathcal{C}, \mathcal{C}')] p_t(\mathcal{C}') \qquad (8)$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\langle Y_t^2\rangle = 2\sum_{\mathcal{C},\mathcal{C}'} [M_1(\mathcal{C}, \mathcal{C}') - M_{-1}(\mathcal{C}, \mathcal{C}')] q_t(\mathcal{C}') + \sum_{\mathcal{C},\mathcal{C}'} [M_1(\mathcal{C}, \mathcal{C}') + M_{-1}(\mathcal{C}, \mathcal{C}')] p_t(\mathcal{C}') \qquad (9)$$

where

$$p_t(\mathcal{C}) = \sum_Y P_t(\mathcal{C}, Y) \qquad q_t(\mathcal{C}) = \sum_Y Y P_t(\mathcal{C}, Y).$$

It has been shown [35, 36] that in the long time limit $p_t(\mathcal{C})$ and $q_t(\mathcal{C})$ have the following asymptotic behaviour:

$$p_t(\mathcal{C}) \to p(\mathcal{C}) \qquad q_t(\mathcal{C}) - Jtp(\mathcal{C}) \to r(\mathcal{C}). \qquad (10)$$

If one substitutes the asymptotics (3), (5) and (10) in (8) and (9) one finds

$$J = \sum_{\mathcal{C},\mathcal{C}'} [M_1(\mathcal{C}, \mathcal{C}') - M_{-1}(\mathcal{C}, \mathcal{C}')] p(\mathcal{C}') \qquad (11)$$

which allows one to recover (4), and as

$$\Delta = \lim_{t\to\infty} \left[\frac{\mathrm{d}}{\mathrm{d}t}\langle Y_t^2\rangle - 2\frac{\mathrm{d}\langle Y_t\rangle}{\mathrm{d}t} \sum_{\mathcal{C}} q_t(\mathcal{C})\right] \qquad (12)$$

one finds

$$\Delta = (1+x)\frac{PH}{(P+H)(P+H-1)} + 2\sum_{\mathcal{C},\mathcal{C}'}[M_1(\mathcal{C},\mathcal{C}') - M_{-1}(\mathcal{C},\mathcal{C}')]r(\mathcal{C}') - 2J\sum_{\mathcal{C}} r(\mathcal{C}). \tag{13}$$

Thus to calculate Δ we need to know $r(\mathcal{C})$. Multiplying both sides of equation (7) by Y and summing over Y gives the time evolution of $q_t(\mathcal{C})$. After substituting the asymptotic behaviour (10) in the expression thus obtained, one finds that the $r(\mathcal{C})$ satisfy the following system of linear equations

$$\sum_{\mathcal{C}'} M(\mathcal{C},\mathcal{C}')r(\mathcal{C}') - \left(\sum_{\mathcal{C}'} M(\mathcal{C}',\mathcal{C})\right)r(\mathcal{C}) = Jp(\mathcal{C}) - \sum_{\mathcal{C}'}[M_1(\mathcal{C},\mathcal{C}') - M_{-1}(\mathcal{C},\mathcal{C}')]p(\mathcal{C}') \tag{14}$$

where $M = M_1 + M_0 + M_{-1}$. Hence, in order to compute Δ from (13), one has to solve the system (14). This is done in section 3, where the $r(\mathcal{C})$ solutions of (14) are obtained using a matrix ansatz. Then one has to perform the two sums which appear in (13) and this is done in section 4.

3. The matrix method

The method we use to solve (14) is an extension of what was done in [35]. We associate with each configuration $\mathcal{C}$ a product of operators. For brevity, the product of operators associated with configuration $\mathcal{C}$ is also denoted by $\mathcal{C}$. In the product there are P operators representing particles, which we note by D, and H operators representing holes, which we denote by E. Thus

$$\mathcal{C} = \prod_{i=1}^{L}[\tau_i D + (1-\tau_i)E] \tag{15}$$

with $\tau_i = 0$ if site i is empty or $\tau_i = 1$ if site i is occupied in $\mathcal{C}$.

Assume that the operators D and E satisfy [31, 32, 39, 40]

$$DE - xED = (1-x)(D+E) \tag{16}$$

or if we introduce δ and ϵ such that

$$D = 1+\delta \qquad \text{and} \qquad E = 1+\epsilon \tag{17}$$

then δ and ϵ satisfy the deformed harmonic oscillator algebra [41, 42], namely

$$\delta\epsilon - x\epsilon\delta = 1-x. \tag{18}$$

Using (17) and (18) repetitively, one can always re-order the ϵ and δ and write (15) as

$$\mathcal{C} = \sum_{m,n} A(\mathcal{C};m,n)\epsilon^m\delta^n. \tag{19}$$

This defines the coefficients $A(\mathcal{C};m,n)$. In appendix A we show that if $r(\mathcal{C})$ are given by

$$r(\mathcal{C}) = \sum_{m,n} A(\mathcal{C};m,n)r(m,n) \tag{20}$$

with

$$r(m,n) = 0 \qquad \text{if } n \neq m$$

and

$$r(n,n) = -\binom{P+H}{P}^{-2} \sum_{i=1}^{n} \frac{1}{1-x^i} \tag{21}$$

then $r(\mathcal{C})$ are solutions of (14).

4. Calculation of the diffusion constant

The expression (13) of Δ can be rewritten

$$\Delta = (1+x)\frac{PH}{(P+H)(P+H-1)} + 2\sum_{\mathcal{C}'}[r(E\mathcal{C}'D) - xr(D\mathcal{C}'E)] - 2J\sum_{\mathcal{C}} r(\mathcal{C}) \tag{22}$$

where the sum over $\mathcal{C}'$ is over all the configurations with $P-1$ particles and $H-1$ holes and the sum over $\mathcal{C}$ is over all the configurations with P particles and H holes. The calculation of the two sums which appear in (22) is the last difficulty we have to overcome to obtain Δ.

By subtracting (A9) from (A8) one can see that

$$2[r(E\mathcal{C}'D) - xr(D\mathcal{C}'E)] = (1-x)[r(D\mathcal{C}') + r(\mathcal{C}'D) + r(E\mathcal{C}') + r(\mathcal{C}'E)] - (1+x)p(\mathcal{C})$$

and using this relation into (22) leads to a simpler formula for Δ:

$$\Delta = (1-x)\sum_{\mathcal{C}'}[r(D\mathcal{C}') + r(\mathcal{C}'D) + r(E\mathcal{C}') + r(\mathcal{C}'E)] - 2J\sum_{\mathcal{C}} r(\mathcal{C}). \tag{23}$$

As explained in appendix B, using generating functions and the fact that $r(\mathcal{C})$ is linear, the two sums which appear in (23) can be expressed in terms of the scalars $r((\delta+\epsilon)^{2l})$, and this leads to

$$\Delta = 2(1-x)\sum_{l=0}^{\min(P,H)} \frac{(P+H)!}{(2l)!(P-l)!(H-l)!} \frac{[PH - l(P+H)]}{(P+H)(P+H-1)} r((\delta+\epsilon)^{2l}). \tag{24}$$

Lastly, the $r((\delta+\epsilon)^{2l})$ are shown in appendix C to be given by

$$r((\delta+\epsilon)^{2l}) = -\binom{P+H}{P}^{-2} \sum_{n=1}^{l} \binom{2l}{l+n} \frac{1+x^n}{1-x^n} \tag{25}$$

so that (24) becomes

$$\Delta = \frac{2(1-x)}{L(L-1)}\binom{P+H}{P}^{-2} \sum_{n=1}^{\min(P,H)} \frac{1+x^n}{1-x^n} \sum_{l=n}^{\min(P,H)} \frac{(P+H)![l(P+H)-PH]}{(P-l)!(H-l)!(l-n)!(l+n)!}. \tag{26}$$

If we replace $[l(P+H)-PH]$ by $n^2 + (l-n)(l+n) - (P-l)(H-l)$ in (26), we obtain three terms

$$\begin{aligned}\sum_{l=n}^{\min(P,H)} \frac{(P+H)![l(P+H)-PH]}{(P-l)!(H-l)!(l-n)!(l+n)!} &= n^2 \sum_{l=n}^{\min(P,H)} \frac{(P+H)!}{(P-l)!(H-l)!(l-n)!(l+n)!} \\ &\quad + \sum_{l=n+1}^{\min(P,H)} \frac{(P+H)!}{(P-l)!(H-l)!(l-n-1)!(l+n-1)!} \\ &\quad - \sum_{l=n}^{\min(P,H)} \frac{(P+H)!}{(P-1-l)!(H-1-l)!(l-n)!(l+n)!}.\end{aligned}$$

The last two terms cancel (by just shifting the variable l by one in the last sum); moreover, using the following identity

$$\sum_{l=n}^{\min(P,H)} \frac{(P+H)!}{(P-l)!(H-l)!(l-n)!(l+n)!} = \binom{P+H}{P+n}\binom{P+H}{H+n}$$

(which can be proved by calculating the coefficient of $z^{P-H}y^{-2n}$ in both sides of $(z+z^{-1}+y+y^{-1})^{P+H} = (z+y)^{P+H}(1+z^{-1}y^{-1})^{P+H}$), one ends up with our final expression for Δ

$$\Delta = 2\frac{(1-x)}{L(L-1)} \sum_{n=1}^{\min(P,H)} n^2 \frac{1+x^n}{1-x^n} \frac{(P!)^2(H!)^2}{(P+n)!(P-n)!(H+n)!(H-n)!}$$

(which is identical to (6) with the convention that $(-p)! = \infty$ for $p \geqslant 1$).

5. Scaling form and crossover between the EW and the KPZ equation

From (6), we can find the scaling form of the diffusion constant when the size L of the system becomes large and the asymmetry becomes weak ($x \to 1$). If we write $\rho = P/L$, $1-\rho = H/L$ and $x = \exp(-f)$ with f small (hence $f \sim 1-x$), we obtain with the help of the Stirling formula

$$\Delta \simeq \frac{2f}{L^2} \sum_{n\geqslant 1} n^2 \frac{\cosh nf/2}{\sinh nf/2} \exp\left(\frac{-n^2}{L\rho(1-\rho)}\right). \tag{27}$$

If we choose a scaling such that $f \sim L^{-1/2}$, or more precisely if we define ϕ by

$$\phi = \frac{f\sqrt{L\rho(1-\rho)}}{2} \tag{28}$$

one finds that (27) becomes, in the limit $L \to \infty$, $f \to 0$ with fixed ϕ

$$\Delta \simeq \frac{4\rho(1-\rho)}{L}\phi \int_0^\infty \mathrm{d}y \exp(-y^2) \frac{y^2}{\tanh(\phi y)} \tag{29}$$

and this confirms the scaling form suggested in [43] by a perturbative expansion around $x = 1$.

We are going now to see how our exact result (29) for the exclusion model can be re-expressed in terms of the KPZ equation. The exclusion model can be mapped onto a growth process [44] in $(1+1)$ dimensions: one defines for each site i a height $h_i(t)$ at time t by

$$h_i(t) - h_{i-1}(t) = 1 - 2\tau_i(t) \tag{30}$$

and the stochastic dynamics of the exclusion model induces a growth rule for the heights $h_i(t)$. Namely, when a particle jumps from site i to site $i+1$, the height $h_i(t)$ increases by 2, and when a particle jumps from site $i+1$ to site i, the height $h_i(t)$ decreases by 2. Hence, random jumps of particles mimic a stochastic deposition–evaporation process. With this mapping, the height $h_L(t)$ at time t is the integrated current through the bond $(L, 1)$

$$h_L(t) - h_L(0) = 2Y_t. \tag{31}$$

Due to the conservation of the total number of particles in the exclusion process, we have tilted periodic boundary conditions for the heights (obtained by summing equation (30) on i)

$$h_{i+L}(t) - h_i(t) = (1-2\rho)L = \kappa L \tag{32}$$

where the parameter $\kappa = 1 - 2\rho$ represents the tilt.

In the continuum limit where i becomes a continuous variable z, this growth model is expected to be described by the KPZ equation

$$\frac{\partial h}{\partial t} = \nu \frac{\partial^2 h}{\partial z^2} + \frac{\lambda}{2}\left(\frac{\partial h}{\partial z}\right)^2 + \eta(z,t) \tag{33}$$

with tilted boundary conditions $(h(z+L,t) = h(z,t) + \kappa L)$.

In (33), $\eta(z,t)$ is a Gaussian white noise with zero mean and covariance

$$\langle \eta(z,t)\eta(z',t')\rangle = D\delta(z-z')\delta(t-t'). \tag{34}$$

We recall [45] that in order to be well defined the KPZ equation contains an implicit short length cut-off; usually equation (33) is rewritten in Fourier space and only modes with a wavenumber less than a ultraviolet cut-off are retained. Here we shall take this cut-off as equal to 1.

If the growth model is well described by the KPZ equation [46], one should be able to express the coefficients (D, ν, λ) in (33) in terms of the parameters x and ρ of the exclusion process. This can be done [47] by matching some physical quantities that can be calculated exactly in both models.

For the discrete growth process (30) the stationary distribution of the heights differences is given by (2). Therefore, if $1 \ll j - i \ll L$, we have in the long time limit

$$\langle [h_j - h_i]^2\rangle - \langle h_j - h_i\rangle^2 = 4\rho(1-\rho)(j-i). \tag{35}$$

The fluctuations of the mean height of the interface for the symmetric process $(x = 1)$ are well known [24] and they can be computed from (29) by taking $\phi = 0$. Then

$$\langle [h_L(t) - h_L(0)]^2\rangle - \langle h_L(t) - h_L(0)\rangle^2 = 4[\langle Y_t^2\rangle - \langle Y_t\rangle^2] \simeq \frac{8\rho(1-\rho)}{L} t. \tag{36}$$

Finally, from the formula for the current (4) and (31), the speed of the interface is

$$\frac{1}{t}\langle h_L(t)\rangle \to 2(1-x)\rho(1-\rho)\frac{L}{L-1} = (1-x)\frac{(1-\kappa^2)}{2}\frac{L}{L-1} \tag{37}$$

where κ is defined in (32).

The stationary measure of the KPZ equation is known [25] (it is Gaussian and does not depend on the nonlinearity coefficient λ). Therefore, all the equal-time averages in the stationary state can be computed. For instance, one obtains that in the long time limit the fluctuations of the height difference between two points x and y such that $1 \ll y - x \ll L$ are given by

$$\langle [h(y,t) - h(x,t)]^2\rangle - \langle h(y,t) - h(x,t)\rangle^2 = \frac{D}{2\nu}(y-x). \tag{38}$$

The fluctuations of the mean height of the interface in the linear case $(\lambda = 0)$ can also be explicitly computed by just integrating (33) on the range 0 to L. One obtains

$$\langle [h(L,t) - h(L,0)]^2\rangle - \langle h(L,t) - h(L,0)\rangle^2 \simeq \frac{D}{L} t. \tag{39}$$

The coefficient λ of the nonlinearity is related to the dependence of the growth rate on the tilt [45]; this can be seen by imposing a tilt to the interface $h(z,t) \to h(z,t) + \kappa z$ and computing how the average speed $v_\infty = \langle \partial h/\partial t\rangle$ for an infinite system varies with κ; one obtains that

$$\frac{\partial^2 v_\infty}{\partial \kappa^2}(0) = \lambda. \tag{40}$$

It should be noted that to obtain a finite expression for the speed v_∞ one has to introduce a short length cut-off. However, the second derivative (40) does not depend on this cut-off.

It is now possible to establish a correspondence between the KPZ equation and the exclusion model by comparing expressions (35)–(37) with expressions (38)–(40):

$$\begin{aligned} D &= 8\rho(1-\rho) \\ \nu &= 1 \\ \lambda &= -(1-x). \end{aligned} \tag{41}$$

Remark. It has been shown [48] that the finite size correction to the growth velocity is given by

$$v_L - v_\infty = -\frac{D\lambda}{4\nu L}$$

to the first order in $1/L$ for the KPZ equation. If this finite size correction is extracted from (37) the following relation is obtained

$$-\frac{D\lambda}{4\nu L} = 2(1-x)\rho(1-\rho)\frac{1}{L}$$

which confirms (41).

Let us now consider the fluctuations of the height above site L for the general asymmetric growth process. We define

$$W(L,t) = \langle[h_L(t) - h_L(0)]^2\rangle - \langle h_L(t) - h_L(0)\rangle^2 = 4[\langle Y_t^2\rangle - \langle Y_t\rangle^2]. \tag{42}$$

To compute the same quantity from the KPZ equation, one can use dimensionless variables [3, 49] obtained by rescaling time, space and height as follows:

$$t = \frac{\nu^5}{\lambda^4 D^2}T \qquad z = \frac{\nu^3}{\lambda^2 D}Z \qquad h = \frac{\nu}{\lambda}H. \tag{43}$$

With these rescalings, the KPZ equation does not contain any explicit parameter any more

$$\frac{\partial H}{\partial T} = \frac{\partial^2 H}{\partial Z^2} + \frac{1}{2}\left(\frac{\partial H}{\partial Z}\right)^2 + \eta(Z,T) \tag{44}$$

and $\eta(Z,T)$ is a Gaussian white noise with zero mean and covariance

$$\langle \eta(Z,T)\eta(Z',T')\rangle = \delta(Z-Z')\delta(T-T').$$

It is then possible to write, just by dimensional analysis,

$$W(L,t) = \left(\frac{\nu}{\lambda}\right)^2 w\left(\frac{\lambda^2 DL}{\nu^3}, \frac{\lambda^4 D^2 t}{\nu^5}\right) = \frac{Dt}{L}F\left(\frac{\lambda^2 DL}{\nu^3}, \frac{\nu t}{L^2}\right) \tag{45}$$

where F is a scaling function characteristic of the KPZ equation.

For a finite system, W is linear in time in the long time limit. This means that the function F does not depend on the variable $\nu t/L^2$ any more, so that

$$F\left(\frac{\lambda^2 DL}{\nu^3}, \frac{\nu t}{L^2}\right) \to F(g,\infty) \qquad \text{when } t\to\infty \tag{46}$$

$$\text{with } g = \frac{\lambda^2 DL}{\nu^3}. \tag{47}$$

The scaling function $F(g, \infty)$ measures the fluctuations of the height in the KPZ equation as a function of the dimensionless variable g which characterizes the strength of the nonlinearity in the KPZ equation. Hence, in the long time limit

$$W(L, t) = \frac{Dt}{L} F(g, \infty). \tag{48}$$

Using (41) one finds that

$$g = \frac{\lambda^2 D L}{\nu^3} = 8(1-x)^2 \rho(1-\rho) L = 32\phi^2 \tag{49}$$

where the variable ϕ has been defined in (28). If we compare (49) to (29) we obtain for the scaling function $F(g, \infty)$

$$F(g, \infty) = \frac{\sqrt{g}}{2\sqrt{2}} \int_0^\infty \mathrm{d}y \exp(-y^2) \frac{y^2}{\tanh((\sqrt{g}/\sqrt{32})y)}. \tag{50}$$

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Appendix A. Proof that equations (19), (20) and (21) solve (14)

In this appendix, we show that when δ and ϵ satisfy (18)

$$\delta\epsilon - x\epsilon\delta = 1 - x \tag{A1}$$

$r(\mathcal{C})$ given by equations (19), (20) and (21) are solutions of (14).

Trying to extend the solution given in [35], we make the hypothesis that $r(\mathcal{C})$ will be linear functionals on the algebra generated by 1 (the identity operator), D and E (or equivalently by 1, δ and ϵ). This means that for arbitrary numbers λ_1 and λ_2 and for arbitrary operators B_1 and B_2 (consisting of sums of products of Ds and Es), one has

$$r(\lambda_1 B_1 + \lambda_2 B_2) = \lambda_1 r(B_1) + \lambda_2 r(B_2).$$

As a consequence, one has from (16)

$$r(B_1 DEB_2) - xr(B_1 EDB_2) = (1-x)[r(B_1 DB_2) + r(B_1 EB_2)]. \tag{A2}$$

Another consequence of the fact that $r(\mathcal{C})$ are linear is that if

$$\mathcal{C} = \sum_{m,n} A(\mathcal{C}; m, n)\epsilon^m \delta^n \tag{A3}$$

then

$$r(\mathcal{C}) = \sum_{m,n} A(\mathcal{C}; m, n) r(\epsilon^m \delta^n). \tag{A4}$$

Note that the quantities $a_k(P, H)$ defined by

$$a_k(P, H) = \sum_{m,n} A(\mathcal{C}; m, n)\delta_{m-n,k} \tag{A5}$$

do not depend on $\mathcal{C}$ itself but depend only on the number P of particles and H of holes in $\mathcal{C}$:

$$a_k(P, H) = \binom{H+P}{H-k}. \tag{A6}$$

(This can be easily shown by choosing $D = 1 + z^{-1}$ and $E = 1 + z$ where z is a number, that is $\delta = z^{-1}$ and $\epsilon = z$. For this very simple choice, δ and ϵ satisfy (A1) and, as with this choice D and E commute, (A3) becomes

$$(1 + z^{-1})^P (1 + z)^H = \sum_{m,n} A(\mathcal{C}; m, n) z^{m-n}$$

which leads immediately to (A6).)

By using the algebra (16) and (A2) the system (14) that we have to solve reduces to one of the following four cases depending on the occupation numbers of the two sites surrounding the marked bond (the rule (A2) generates simplifications in the bulk similar to those described in [32]):

- For $\tau_1 = 1$ and $\tau_L = 1$ (i.e. $\mathcal{C} = D\mathcal{C}'D$, with $P - 2$ particles and H holes in $\mathcal{C}'$)

$$Jp(\mathcal{C}) = (1 - x)[r(D\mathcal{C}') - r(\mathcal{C}'D)]. \tag{A7}$$

- For $\tau_1 = 1$ and $\tau_L = 0$ (i.e. $\mathcal{C} = D\mathcal{C}'E$, with $P - 1$ particles and $H - 1$ holes in $\mathcal{C}'$)

$$Jp(D\mathcal{C}'E) - p(E\mathcal{C}'D) = r(E\mathcal{C}'D) - xr(D\mathcal{C}'E) - (1 - x)[r(D\mathcal{C}') + r(\mathcal{C}'E)]. \tag{A8}$$

- For $\tau_1 = 0$ and $\tau_L = 1$ (i.e. $\mathcal{C} = E\mathcal{C}'D$, with $P - 1$ particles and $H - 1$ holes in $\mathcal{C}'$)

$$Jp(E\mathcal{C}'D) + xp(D\mathcal{C}'E) = xr(D\mathcal{C}'E) - r(E\mathcal{C}'D) + (1 - x)[r(E\mathcal{C}') + r(\mathcal{C}'D)]. \tag{A9}$$

- For $\tau_1 = 0$ and $\tau_L = 0$ (i.e. $\mathcal{C} = E\mathcal{C}'E$, with P particles and $H - 2$ holes in $\mathcal{C}'$)

$$Jp(\mathcal{C}) = (1 - x)[r(\mathcal{C}'E) - r(E\mathcal{C}')]. \tag{A10}$$

A simple consequence of (A1) is that for $m \geqslant 1$

$$\delta\epsilon^m - x^m\epsilon^m\delta = (1 - x^m)\epsilon^{m-1} \qquad \delta^m\epsilon - x^m\epsilon\delta^m = (1 - x^m)\delta^{m-1}. \tag{A11}$$

Then if one defines

$$s(m, n) = r(\epsilon^m\delta^n) - r(\epsilon^{m+1}\delta^{n+1}) \tag{A12}$$

one obtains by substituting (A3) and (A4) into (A7)–(A10), and by using (2)

- for $\mathcal{C}'$ having $P - 2$ particles and H holes

$$J\binom{P+H}{P}^{-1} = (1 - x)\sum_{m,n} A(\mathcal{C}'; m, n)(1 - x^m)s(m - 1, n) \tag{A13}$$

- for $\mathcal{C}'$ having $P - 1$ particles and $H - 1$ holes

$$\begin{aligned}(J - 1)\binom{P+H}{P}^{-1} = -\sum_{m,n} A(\mathcal{C}'; m, n)\{&(1 - x^{m+n+2})s(m, n) + (1 - x^m)s(m - 1, n) \\ &+ (1 - x^n)s(m, n - 1) + x(1 - x^m)(1 - x^n)s(m - 1, n - 1)\}\end{aligned} \tag{A14}$$

- for $\mathcal{C}'$ having $P - 1$ particles and $H - 1$ holes

$$\begin{aligned}(J + x)\binom{P+H}{P}^{-1} = \sum_{m,n} A(\mathcal{C}'; m, n)\{&(1 - x^{m+n+2})s(m, n) + x(1 - x^m)s(m - 1, n) \\ &+ x(1 - x^n)s(m, n - 1) + x(1 - x^m)(1 - x^n)s(m - 1, n - 1)\}\end{aligned} \tag{A15}$$

- for $\mathcal{C}'$ having P particles and $H - 2$ holes

$$J\binom{P+H}{P}^{-1} = (1 - x)\sum_{m,n} A(\mathcal{C}'; m, n)(1 - x^n)s(m, n - 1). \tag{A16}$$

We are now going to show that the choice (21) for $r(m,n) \equiv r(\epsilon^m \delta^n)$ does solve (A13)–(A16). From (A12) one obtains

$$s(m,n) = r(m,n) - r(m+1,n+1) = \binom{P+H}{P}^{-2} \frac{1}{1-x^{n+1}} \delta_{n,m} \qquad \text{(A17)}$$

and (A13)–(A16) become using (A5)

$$J \binom{P+H}{P} = (1-x) a_1(P-2, H)$$

$$(J-1) \binom{P+H}{P} = -(1+x) a_0(P-1, H-1) - a_1(P-1, H-1) - a_{-1}(P-1, H-1)$$

$$(J+x) \binom{P+H}{P} = (1+x) a_0(P-1, H-1) + x a_1(P-1, H-1) + x a_{-1}(P-1, H-1)$$

$$J \binom{P+H}{P} = (1-x) a_{-1}(P, H-2).$$

These equalities follow easily when the explicit expressions (4) and (A6) of J and of the a_k are used. This proves that equations (19), (20) and (21) solve (14).

Appendix B. Calculation of the two sums which appear in (23)

The sum $\sum_{\mathcal{C}} r(\mathcal{C})$ over all the configurations with fixed numbers P of particles and H of holes is the coefficient of $\lambda^P \mu^H$ in the generating function $r((\lambda D + \mu E)^{P+H})$. As $D = 1+\delta$ and $E = 1+\epsilon$, one can write

$$r((\lambda D + \mu E)^{P+H}) = \sum_{k=0}^{P+H} \binom{P+H}{k} (\lambda+\mu)^{P+H-k} r((\lambda\delta + \mu\epsilon)^k). \qquad \text{(B1)}$$

From (21) where the expressions of $r(m,n) = r(\epsilon^m \delta^n)$ are given, we know that the only non-zero terms in $r((\lambda\delta + \mu\epsilon)^k)$ are those that contain the same number of δ and of ϵ. Consequently, in (B1), the only non-zero terms correspond to k even. Therefore, for $k = 2l$ one has $r((\lambda\delta + \mu\epsilon)^{2l}) = (\lambda\mu)^l r((\delta+\epsilon)^{2l})$.

Formula (B1) then becomes

$$r((\lambda D + \mu E)^{P+H}) = \sum_{l=0}^{E[(P+H)/2]} \binom{P+H}{2l} (\lambda+\mu)^{P+H-2l} (\lambda\mu)^l r((\delta+\epsilon)^{2l}) \qquad \text{(B2)}$$

where $E[(P+H)/2]$ is the integer part of $(P+H)/2$. From (B2) the coefficient of $\lambda^P \mu^H$ can be extracted easily:

$$\sum_{\mathcal{C}} r(\mathcal{C}) = \sum_{l=0}^{\min(P,H)} \binom{P+H}{2l} \binom{P+H-2l}{P-l} r((\delta+\epsilon)^{2l}). \qquad \text{(B3)}$$

In a similar manner, the term $\sum_{\mathcal{C}'} [r(D\mathcal{C}') + r(\mathcal{C}'D) + r(E\mathcal{C}') + r(\mathcal{C}'E)]$ is the coefficient of $\lambda^{P-1}\mu^{H-1}$ in the generating function

$$r((\lambda D + \mu E)^{P+H-2}(D+E)) + r((D+E)(\lambda D + \mu E)^{P+H-2}).$$

This expression can be written

$$\sum_{k=0}^{P+H-2} \binom{P+H-2}{k} (\lambda+\mu)^{P+H-2-k}(4r((\lambda\delta+\mu\epsilon)^k) + r((\lambda\delta+\mu\epsilon)^k(\delta+\epsilon)) + r((\delta+\epsilon)(\lambda\delta+\mu\epsilon)^k)). \tag{B4}$$

The coefficient of $\lambda^{P-1}\mu^{H-1}$ can be extracted from the first term on the right-hand side of (B4) exactly as above. As for the two remaining terms, we see that the only non-zero terms in $r((\lambda\delta+\mu\epsilon)^k(\delta+\epsilon))$ and in $r((\delta+\epsilon)(\lambda\delta+\mu\epsilon)^k)$ correspond to k odd; one has for $k = 2l-1$, using the same argument as above

$$r((\lambda\delta+\mu\epsilon)^{2l-1}(\delta+\epsilon)) = \lambda^{l-1}\mu^l r((\delta+\epsilon)^{2l-1}\delta) + \lambda^l\mu^{l-1} r((\delta+\epsilon)^{2l-1}\epsilon) \tag{B5}$$

and

$$r((\delta+\epsilon)(\lambda\delta+\mu\epsilon)^{2l-1}) = \lambda^{l-1}\mu^l r(\delta(\delta+\epsilon)^{2l-1}) + \lambda^l\mu^{l-1} r(\epsilon(\delta+\epsilon)^{2l-1}). \tag{B6}$$

We now use the identities

$$r(\epsilon(\delta+\epsilon)^{2l-1}) = r((\delta+\epsilon)^{2l-1}\delta) \tag{B7}$$

$$r(\delta(\delta+\epsilon)^{2l-1}) = r((\delta+\epsilon)^{2l-1}\epsilon) \tag{B8}$$

(which follow from the fact that using (18) one can always write $(\delta+\epsilon)^k = \sum_{m,n} a(k;m,n)\epsilon^m\delta^n$ with symmetric coefficients $a(k;m,n)$, that is $a(k;m,n) = a(k;n,m)$) and rewrite the last two terms in (B4) as

$$r((\lambda\delta+\mu\epsilon)^{2l-1}(\delta+\epsilon)) + r((\delta+\epsilon)(\lambda\delta+\mu\epsilon)^{2l-1}) = (\lambda^{l-1}\mu^l + \lambda^l\mu^{l-1}) r((\delta+\epsilon)^{2l}). \tag{B9}$$

Now the calculation can readily be completed and we obtain

$$\sum_{\mathcal{C}'}[r(D\mathcal{C}') + r(\mathcal{C}'D) + r(E\mathcal{C}') + r(\mathcal{C}'E)] = \sum_{l=0}^{\min(P-1,H-1)} \left\{ 4\binom{P+H-2}{2l}\binom{P+H-2l-2}{P-1-l} + \binom{P+H-2}{2l-1}\binom{P+H-2l}{P-l} \right\} r((\delta+\epsilon)^{2l}). \tag{B10}$$

Substituting (B3) and (B10) in (23) leads to the expression (24) of the diffusion constant in terms of the scalars $r((\delta+\epsilon)^{2l})$.

Appendix C. Proof of identity (25)

From the deformed harmonic oscillator algebra (18)

$$\delta\epsilon - x\epsilon\delta = 1 - x \tag{C1}$$

and its consequences (A11), one can show by recursion that $(\delta+\epsilon)^n$ can be written as

$$(\epsilon+\delta)^n = \sum_{p=0}^{E[n/2]} Q^n_{n-2p}(x) \sum_{i=0}^{n-2p} \begin{bmatrix} n-2p \\ i \end{bmatrix} \epsilon^i \delta^{n-2p-i} \tag{C2}$$

(by re-ordering ϵ and δ) where $E[n/2]$ is the integer part of $n/2$, the polynomials Q^n_k satisfy the recursion formula

$$Q^{n+1}_k(x) = Q^n_{k-1}(x) + (1 - x^{k+1}) Q^n_{k+1}(x)$$

(with $Q_k^0 = \delta_{k,0}$) and the deformed binomial coefficients (called Gaussian binomials) are defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-x^n)(1-x^{n-1})\dots(1-x^{n-k+1})}{(1-x^k)(1-x^{k-1})\dots(1-x)}.$$

These Gaussian binomials satisfy identities similar to those of usual binomials

$$\begin{bmatrix} n+1 \\ k \end{bmatrix} = x^k \begin{bmatrix} n \\ k \end{bmatrix} + \begin{bmatrix} n \\ k-1 \end{bmatrix}$$

$$\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n \\ n-k \end{bmatrix}.$$

We adopt the convention that $[{n \atop k}] = 0$ whenever $n < 0$ or $k > n$.

Formula (C2) is valid for any choice of δ and ϵ that satisfies the algebra (C1). In particular, one can choose $\epsilon = z$ and $\delta = 1/z$ where z is a real variable. Then (C2) reduces to

$$(z + 1/z)^n = \sum_{p=0}^{E[n/2]} Q_{n-2p}^n(x) \sum_{i=0}^{n-2p} \begin{bmatrix} n-2p \\ i \end{bmatrix} z^{2i+2p-n} \qquad \text{(C3)}$$

and by identifying the coefficients of the powers of z on both sides of (C3) one obtains

$$\binom{n}{j} = \sum_{p=0}^{E[n/2]} Q_{n-2p}^n(x) \begin{bmatrix} n-2p \\ j-p \end{bmatrix} \qquad \text{for } 0 \leqslant j \leqslant n. \qquad \text{(C4)}$$

If n is even ($n = 2l$), the preceeding formula becomes

$$\binom{2l}{l+j} = \sum_{p=j}^{l} Q_{2p}^{2l}(x) \begin{bmatrix} 2p \\ p+j \end{bmatrix} \qquad \text{for } 0 \leqslant j \leqslant l. \qquad \text{(C5)}$$

We can now use the ordering identity (C2) to compute $r((\epsilon + \delta)^{2l})$

$$r((\epsilon+\delta)^{2l}) = \sum_{p=0}^{l} Q_{2p}^{2l}(x) \sum_{i=0}^{2p} \begin{bmatrix} 2p \\ i \end{bmatrix} r(\epsilon^i \delta^{2p-i}) \qquad \text{(C6)}$$

which becomes using (21)

$$r((\epsilon+\delta)^{2l}) = \frac{-1}{\binom{P+H}{P}^2} \sum_{p=1}^{l} Q_{2p}^{2l}(x) \begin{bmatrix} 2p \\ p \end{bmatrix} \sum_{j=1}^{p} \frac{1}{1-x^j}. \qquad \text{(C7)}$$

Our goal is to prove (25)

$$r((\delta+\epsilon)^{2l}) = \frac{-1}{\binom{P+H}{P}^2} \sum_{j=1}^{l} \frac{1+x^j}{1-x^j} \binom{2l}{l+j} \qquad \text{(C8)}$$

which can be rewritten using (C5) as

$$\begin{aligned} r((\delta+\epsilon)^{2l}) &= \frac{-1}{\binom{P+H}{P}^2} \sum_{j=1}^{l} \frac{1+x^j}{1-x^j} \sum_{p=j}^{l} Q_{2p}^{2l}(x) \begin{bmatrix} 2p \\ p+j \end{bmatrix} \\ &= \frac{-1}{\binom{P+H}{P}^2} \sum_{p=1}^{l} Q_{2p}^{2l}(x) \sum_{j=1}^{p} \begin{bmatrix} 2p \\ p+j \end{bmatrix} \frac{1+x^j}{1-x^j}. \end{aligned} \qquad \text{(C9)}$$

So far we have proved (C7) and we would like to prove (C9). We are now going to show that (C7) and (C9) are indeed the same due to the following identity on deformed binomials

$$\begin{bmatrix} 2p \\ p \end{bmatrix} \sum_{j=1}^{p} \frac{1}{1-x^j} = \sum_{j=1}^{p} \begin{bmatrix} 2p \\ p+j \end{bmatrix} \frac{1+x^j}{1-x^j}. \tag{C10}$$

This formula can be established by induction on p. The case $p = 1$ can be checked directly. If the formula is true for p, we are going to show that it remains true for $p+1$:

$$\begin{bmatrix} 2p+2 \\ p+1 \end{bmatrix} \sum_{j=1}^{p+1} \frac{1}{1-x^j} = \sum_{j=1}^{p+1} \begin{bmatrix} 2p+2 \\ p+1+j \end{bmatrix} \frac{1+x^j}{1-x^j}. \tag{C11}$$

Let us multiply (C11) by the factor $(1-x^{2p+2})(1-x^{2p+1})/(1-x^{p+1})(1-x^{p+1})$ and substract (C10) from it. The left-hand side of (C11) gives

$$\begin{bmatrix} 2p+2 \\ p+1 \end{bmatrix} \frac{1}{1-x^{p+1}} \tag{C12}$$

whereas on the right-hand side one obtains

$$\frac{(1+x^{p+1})}{(1-x^{p+1})} + \sum_{j=1}^{p} \left(\begin{bmatrix} 2p+2 \\ p+1+j \end{bmatrix} - \begin{bmatrix} 2p \\ p+j \end{bmatrix} \frac{(1-x^{2p+2})(1-x^{2p+1})}{(1-x^{p+1})(1-x^{p+1})} \right) \frac{1+x^j}{1-x^j}. \tag{C13}$$

The equality between (C12) and (C13) then follows as a consequence of the identity

$$\left(\begin{bmatrix} 2p+2 \\ p+1+j \end{bmatrix} - \begin{bmatrix} 2p \\ p+j \end{bmatrix} \frac{(1-x^{2p+2})(1-x^{2p+1})}{(1-x^{p+1})(1-x^{p+1})} \right) \frac{1+x^j}{1-x^j}$$
$$= \left(\begin{bmatrix} 2p+1 \\ p+j \end{bmatrix} - \begin{bmatrix} 2p+1 \\ p+j+1 \end{bmatrix} \right) \frac{1+x^{p+1}}{1-x^{p+1}}$$

and this completes the proof of (25).

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